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A LEAST ELEMENT THEORY OF SOLVING LINEAR COMPLEMENTARITY PROBLEM--ETC(U)  
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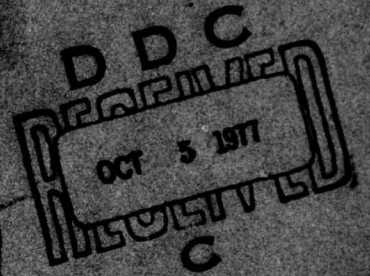
A LEAST ELEMENT THEORY  
OF SOLVING LINEAR COMPLEMENTARITY PROBLEMS  
AS LINEAR PROGRAMS

BY

RICHARD W. COTTLE and JONG-SHI PANG

TECHNICAL REPORT 76-30

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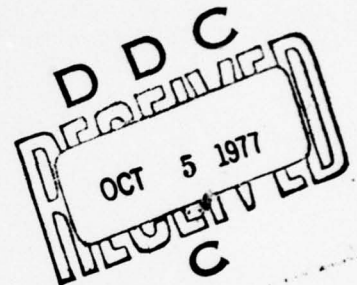
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A LEAST ELEMENT THEORY  
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ABSTRACT

In a previous report [2], the authors have established a least-element interpretation to Mangasarian's theory [5], [6] of formulating some linear complementarity problems as linear programs. In the present report, we extend our previous analysis to a more general class of linear complementarity problems investigated in Mangasarian [7]. Our purposes are (1) to demonstrate how solutions to these problems can be generated from least elements of polyhedral sets and (2) to investigate how these "least-element solutions" are related to the solutions obtained by the linear programming approach as proposed by Mangasarian.

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1. INTRODUCTION

In this paper, we study the linear complementarity problem of finding a vector  $x \in R^n$  satisfying

$$(1.1) \quad x \geq 0, \quad q + Mx \geq 0 \quad \text{and} \quad x^T(q + Mx) = 0$$

where the given  $n$ -vector  $q$  and  $n \times n$  matrix  $M$  satisfy the following three assumptions:

$$(M1) \quad MX = Y + qc^T$$

$$(M2) \quad r^T X + s^T Y \geq 0$$

$$(M3) \quad r^T(X+C) + s^T(Y+C) > 0$$

where  $X$  and  $Y$  are suitable  $Z$ -matrices (i.e., real square matrices whose off-diagonal entries are non-positive),  $C$  is a diagonal matrix whose diagonal elements are the components of the vector  $c$ , and  $r$ ,  $s$  and  $c$  are some non-negative vectors. We denote problem (1.1) by the pair  $(q, M)$ . Its feasible set is defined as the polyhedral set

$$X(q, M) = \{x \in R^n : x \geq 0, \quad q + Mx \geq 0\}.$$

The linear complementarity problem  $(q, M)$  with the vector  $q$  and matrix  $M$  satisfying conditions (M1) - (M3) has recently been studied by Mangasarian [7] who shows that such a problem can be formulated as the linear program

$$(1.2) \quad \text{minimize } p^T x \text{ subject to } x \geq 0 \text{ and } q + Mx \geq 0$$

where  $p = r + M^T s$ . Our purpose in this report is to show that this result is related to a theory of polyhedral sets having least elements. (A vector  $\bar{x}$  belonging to a set  $S \subseteq R^n$  is said to be the least (greatest) element of  $S$  if  $\bar{x} \leq (\geq) x$  for every  $x \in S$ .) The method of derivation used by Mangasarian is not based on least-element arguments. In a previous report [2], the authors have applied this theory of polyhedral sets having least elements to the particular case  $c = 0$  and established that for every  $n$ -vector  $q$ , the linear complementarity problem  $(q, M)$  has a solution which can be generated from the least element of a polyhedral set, thus providing a least-element interpretation to the linear programming formulation of the problem  $(q, M)$  which was initially obtained by Mangasarian in [5]. In the present report, we extend our previous analysis to the general case where  $c$  is merely non-negative, as described at the beginning of the introduction. Our purposes are (1) to establish the least-element characterization of a solution to the linear complementarity problem under consideration and (2) to demonstrate how this "least-element solution" is related to the solution(s) obtained by the linear programming approach as proposed by Mangasarian. Here, we should point out that all

the linear complementarity problems, which are listed in Table 1 of [7] to be solvable as linear programs, satisfy conditions (M1) - (M3). Therefore the least-element theory developed in the present report is applicable to all of them. However, it may not necessarily be applicable to those problems satisfying the more general conditions in Theorem 1 of [7].

It would be appropriate for us to review some of the essential results obtained in [2] for the particular case  $c = 0$ . Using the same notations, we denote problem (1.2) by the triple  $(p, q, M)$  and by  $C$  the class of square matrices  $M$  for which there exist Z-matrices  $X$  and  $Y$  such that the following two conditions are satisfied

$$(C1) \quad MX = Y$$

$$(C2) \quad r^T X + s^T Y > 0 \text{ for some } r, s \geq 0.$$

These are precisely conditions (M1) - (M3) with  $c = 0$ . The following proposition is an immediate consequence of the well-known theorem of Kuhn-Fourier [4] on the solvability of a system of linear relations.

Proposition 1.1. Let  $X$  and  $Y$  be  $n \times n$  matrices. Then the following are equivalent

$$(C2) \quad r^T X + s^T Y > 0 \text{ for some } r, s \geq 0$$

$$(C2)' \quad \left. \begin{array}{l} u \geq 0 \\ Xu \leq 0 \\ Yu \leq 0 \end{array} \right\} \Rightarrow u = 0.$$

We have established useful necessary and sufficient conditions for two Z-matrices  $X$  and  $Y$  to satisfy condition (C2). These are stated below.



Lemma 1.2. Let  $X$  and  $Y$  be  $Z$ -matrices of the same order. Then (C2) holds if and only if there exist a principal rearrangement with permutation matrix  $P$  and a partitioning of  $X$  and  $Y$  such that

$$(1.3a) \quad P^T X P = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad P^T Y P = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

$$(1.3b) \quad \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \text{ is a K-matrix}^*.$$

Using this lemma, we have given necessary and sufficient conditions for  $M \in \mathcal{C}$ .

Theorem 1.3. Let  $M$ ,  $X$  and  $Y$  be  $n \times n$  matrices with  $X$  and  $Y$  both  $Z$ -matrices. Then

$$(C1) \quad MX = Y$$

$$(C2) \quad r^T X + s^T Y > 0 \quad \text{for some } r, s \geq 0$$

if and only if there is a principal rearrangement and partitioning of  $M$ ,  $X$  and  $Y$  such that

$$(1.4a) \quad \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

$$(1.4b) \quad \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \text{ is a K-matrix}$$

$$(1.4c) \quad X \text{ is nonsingular.}$$

---

\* A K-matrix, also known as Minkowski matrix, is a  $Z$ -matrix with a non-negative inverse.

The above-mentioned least-element result for the linear complementarity problem  $(q, M)$  with  $M \in \mathcal{C}$  is stated in the theorem below.

Theorem 1.4. Let  $M \in \mathcal{C}$  and let  $X$  and  $Y$  be  $Z$ -matrices satisfying (C1) and (C2). Suppose  $(q, M)$  is feasible, i.e.  $X(q, M) \neq \emptyset$ . Then the polyhedral set

$$V = \{v \in \mathbb{R}^n : Xv \geq 0, q + Yv \geq 0\}$$

contains a least element  $\bar{v}$ . Moreover, the vector  $\bar{x} = X\bar{v}$  solves the problem  $(q, M)$ .

As a consequence to this theorem, we deduced that for every vector  $q \in \mathbb{R}^n$ , the linear complementarity problem  $(q, M)$  with  $M \in \mathcal{C}$  can be solved as the linear program  $(p, q, M)$  where the vector  $p$  is the (unique) solution to the system of equations

$$p^T X = f$$

for some positive vector  $f$ . We have also shown that the vector  $p$  required in Mangasarian's theory can be obtained in precisely the same way. In the last part of the report, we established several related matrix-theoretic results, and demonstrated that  $\mathcal{C}$  includes all the matrices investigated by Mangasarian in [6].

We explain the notations used in the paper. All vectors and matrices under consideration are real. A  $Z$ -matrix  $X$  is said to be a  $K_0$ -matrix if  $(X + \epsilon I)$  is a  $K$ -matrix for every  $\epsilon > 0$ . The letters  $Z$ ,  $K_0$  and  $K$  will also denote the class of  $Z$ -,  $K_0$ - and  $K$ -matrices respectively.

Various characterizations of  $K$ - and  $K_0$ -matrices can be found in [3].

We denote the range space of a matrix  $A$  by  $\mathcal{R}(A)$ , i.e.  $\mathcal{R}(A)$  consists of those vectors which can be represented as linear combinations of the columns of  $A$ . Let  $M$  be an  $n \times n$  matrix. If  $I, J \subseteq \{1, \dots, n\}$ , we define

$$M_{IJ} = \begin{bmatrix} m_{i_1 j_1} & \dots & m_{i_1 j_t} \\ \vdots & & \vdots \\ m_{i_s j_1} & \dots & m_{i_s j_t} \end{bmatrix}$$

where  $I = \{i_1, \dots, i_s\}$  and  $J = \{j_1, \dots, j_t\}$  with  $1 \leq i_1 < \dots < i_s \leq n$  and  $1 \leq j_1 < \dots < j_t \leq n$ . In particular,  $M_{II}$  is a principal submatrix of  $M$ . Similarly, if  $q \in \mathbb{R}^n$ , we define  $q_I = (q_{i_1}, \dots, q_{i_s})^T$ . We denote the unit vector  $(1, \dots, 1)^T$  by  $e$ .



## 2. CONNECTION WITH LEAST ELEMENTS

2.1. General discussion. Let  $q$  be an  $n$ -vector and  $M$  an  $n \times n$  matrix satisfying conditions (M1) - (M3) for some  $Z$ -matrices  $X$  and  $Y$ , and some non-negative vectors  $r$ ,  $s$  and  $c$ . Throughout this section, these vectors and matrices are assumed to possess the properties just mentioned. We shall develop a least-element study of the linear complementarity problem  $(q, M)$  with such a vector  $q$  and matrix  $M$ . As a consequence of our investigation, we shall establish a least-element interpretation for the result obtained by Mangasarian [7] of formulating such a linear complementarity problem as the linear program  $(p, q, M)$  with  $p = r + M^T s$ . We start by proving a lemma which strengthens condition (M3).

Lemma 2.1. Let  $c$ ,  $r$  and  $s$  be non-negative vectors and let  $X$  and  $Y$  be  $Z$ -matrices. If condition (M3) holds, then

$$(M3)' \quad r + s > 0.$$

Proof: It suffices to show that for every  $i = 1, \dots, n$ ,  $r_i = 0$  ( $s_i = 0$ )  $\Rightarrow s_i > 0$  ( $r_i > 0$ ). So assume  $r_i = 0$ , say. Then

$$\begin{aligned} 0 &< (r^T(X + C) + s^T(Y + C))_i \\ &= \sum_{j \neq i} r_j X_{ji} + \sum_{j \neq i} s_j Y_{ji} + s_i(Y_{ii} + c_i) \\ &\leq s_i(Y_{ii} + c_i). \end{aligned}$$

Thus  $s_i > 0$ . Similarly, we may deduce  $[s_i = 0 \Rightarrow r_i > 0]$ . Therefore (M3)' follows.  $\square$

Remark. Condition (M3)' was referred to as a special case, but not as a consequence of (M3) in [7]. In fact, if  $c > 0$  (as in Corollary 1 of [7]), the two conditions are equivalent.

We recall that if  $M \in \mathbb{C}$ , the matrix  $X$  satisfying (C1) and (C2) must be nonsingular (Theorem 1.3). The following example illustrates that there can be singular matrix  $X$  satisfying (M1) - (M3).

Example 2.2. Let  $q = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$  and  $M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . The problem  $(q, M)$  has two solutions, namely,  $\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$  and  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ . If  $X = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 1 & 0 \\ 0 & -1/2 \end{pmatrix}$ ,  $r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $c = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ , then conditions (M1) - (M3) are satisfied.

Nevertheless,  $X$  is singular. Moreover, it is not hard to verify that  $M \notin \mathbb{C}$ .  $\square$

The fact that there exist such singular matrices  $X$  indicates that in order to develop a least-element theory for the linear complementarity problem  $(q, M)$ , one should not merely concentrate on the range space of  $X$ . In fact, the same example above shows that  $\mathcal{R}(X) \cap X(q, M) = \emptyset$ . Later in our discussion, we will see that this latter relation always holds if  $X$  is singular and the stronger condition (C2) is imposed (see Proposition 2.26 and the remark following it).

It is clear that if  $M \in \mathbb{C}$ , then conditions (M1) - (M3) are satisfied for every vector  $q$ . Nevertheless, if a matrix  $M$  satisfies (M1) - (M3) for some vector  $q$ , it does not necessarily follow that  $M \in \mathbb{C}$ . Example 2.2 illustrates this fact. The following provides another example.

Example 2.3. Let  $q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $M = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$ . Then conditions (M1) - (M3) are satisfied with  $r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ,  $c = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ ,  $X = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$  and  $Y = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 0 \end{pmatrix}$ .

We show  $M \notin \mathcal{C}$ . Suppose not, then there exists  $\begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix} \in Z$  such that

$$\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} -x_{11} - x_{21} & x_{12} + x_{22} \\ 2x_{11} + x_{21} & -2x_{12} - x_{22} \end{pmatrix} \in Z$$

$$\text{and } (r_1, r_2) \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix} + (s_1, s_2) \begin{pmatrix} -x_{11} - x_{21} & x_{12} + x_{22} \\ 2x_{11} + x_{21} & -2x_{12} - x_{22} \end{pmatrix} > 0 \text{ for}$$

some non-negative scalars  $r_1, r_2, s_1$  and  $s_2$ . It follows that  $2x_{11} + x_{21} \leq 0$

which implies  $x_{11} \leq -\frac{1}{2}x_{21} \leq 0$ . Similarly,  $x_{12} + x_{22} \leq 0$  implies

$x_{22} \leq -x_{12} \leq 0$ . Therefore Lemma 1.2 implies that

$$\begin{pmatrix} -x_{11} - x_{21} & x_{12} + x_{22} \\ 2x_{11} + x_{21} & -2x_{12} - x_{22} \end{pmatrix} \in K.$$

In particular,  $\det \begin{pmatrix} -x_{11} - x_{21} & x_{12} + x_{22} \\ 2x_{11} + x_{21} & -2x_{12} - x_{22} \end{pmatrix} > 0$ . Since  $\det M < 0$ , it

follows that

$$(2.1) \quad \det \begin{pmatrix} x_{11} & -x_{12} \\ -x_{21} & x_{22} \end{pmatrix} = x_{11}x_{22} - x_{12}x_{21} < 0.$$

Furthermore,

$$x_{11} + x_{12} < 0 \Rightarrow x_{11} < -x_{21} \leq 0.$$

We have shown that  $x_{22} \leq -x_{12} \leq 0$ ; thus  $x_{11}x_{22} \geq x_{21}x_{12}$  contradicting

(2.1). Therefore  $\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \notin \mathcal{C}$ .  $\square$



The above examples illustrate that the present class of linear complementarity problems  $(q, M)$  with  $q$  and  $M$  satisfying (M1) - (M3) is a genuine extension of the previous class of problems  $(q, M)$  with  $M \in \mathbb{C}$ . Example 2.3 was used originally in [7] by Mangasarian for another purpose.

2.2. The set  $\mathcal{U}$ . We define

$$\mathcal{U} = \{u \in \mathbb{R}^n : u \geq 0, Xu \leq 0, Yu \leq 0, c^T u = 1\}.$$

This polyhedral set  $\mathcal{U}$  plays a very important role throughout our whole discussion. It may be empty, for example, if  $c = 0$  as in the case of  $M \in \mathbb{C}$ . The following example illustrates that it can sometimes be non-empty as well.

Example 2.4. Let  $M = \begin{pmatrix} 0 & 3 & 4 \\ 1 & -1 & 0 \\ 0 & -1 & -3 \end{pmatrix}$  and  $q = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ . Then with  $c = s = e$ ,

$r = 0$ ,  $X = -I$  and  $Y = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ , conditions (M1) - (M3) are

satisfied. In this case,  $u = \begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix} \in \mathcal{U}$ . We shall say more about this example later.  $\square$

The proposition below describes the relationship between vectors in  $\mathcal{U}$  (if any) and solutions to the linear complementarity problem  $(q, M)$ .

Proposition 2.5. If  $u \in \mathcal{U}$ , then  $x = -Xu$  is a solution to  $(q, M)$ .

Proof: If  $x = -Xu$  where  $u \in \mathcal{U}$ , then  $x \geq 0$ ; moreover,

$$\begin{aligned}
q + Mx &= q - MXu \\
&= q - (Y + qc^T)u \\
&= -Yu \geq 0,
\end{aligned}$$

i.e.  $x \in X(q, M)$ . We also have

$$\begin{aligned}
0 &\leq r^T x + s^T (q + Mx) \\
&= -(r^T X + s^T Y)u \leq 0.
\end{aligned}$$

Therefore, for each  $i = 1, \dots, n$ ,

$$r_i x_i = s_i (q + Mx)_i = 0.$$

If  $x_i > 0$ , say, then  $r_i = 0$ . Condition (M3)' implies  $s_i > 0$  which gives  $(q + Mx)_i = 0$ . Therefore the vector  $x$  defined above solves the problem  $(q, M)$ .  $\square$

Remark 1. In the proof above, the assumption that  $X$  and  $Y$  are Z-matrices is required in order for condition (M3)' to be applicable. Therefore, if condition (M3)' holds by itself (as in Corollary 1 of [7]), then the proposition is valid for any matrices  $X$  and  $Y$  (which are not necessarily Z-matrices) satisfying conditions (M1) and (M2).

Remark 2. The feasibility of the problem  $(q, M)$  is not a requirement, but a consequence of Proposition 2.5.

Proposition 2.5 shows that if the set  $\mathcal{U}$  is nonempty, then a solution to the linear complementarity problem  $(q, M)$  can be obtained by first finding any vector  $u$  in  $\mathcal{U}$ , or equivalently, solving the system of linear inequalities

$$u \geq 0, Xu \leq 0, Yu \leq 0 \text{ and } c^T u = 1$$

and then setting  $x = -Xu$ . The question of whether  $\mathcal{U}$  is nonempty is itself answerable by linear programming. The next proposition shows that if  $\mathcal{U} \neq \emptyset$ , then any vector  $x$  such that  $x = -Xu$  for some  $u \in \mathcal{U}$  actually solves the linear program  $(p, q, M)$  where  $p = r + M^T s$ .

Proposition 2.6. If  $\bar{u} \in \mathcal{U}$ , then  $\bar{x} = -X\bar{u}$  solves the linear program  $(p, q, M)$  with  $p = r + M^T s$ .

Proof: If  $x \in X(q, M)$ , then

$$p^T x = (r^T + s^T M)x \geq -s^T q.$$

On the other hand, the proof of the last proposition shows that

$\bar{x} = -X\bar{u} \in X(q, M)$  if  $\bar{u} \in \mathcal{U}$ . Furthermore, with such an  $\bar{x}$ , we have

$$\begin{aligned} p^T \bar{x} &= -(r^T + s^T M)X\bar{u} \\ &= -(r^T X + s^T Y)\bar{u} - s^T q = -s^T q. \end{aligned}$$

Therefore, any such  $\bar{x}$  solves the linear program  $(p, q, M)$ .  $\square$

Corollary 2.7. If  $M$  is nondegenerate (i.e. every principal submatrix of  $M$  is nonsingular) and if  $\mathcal{U} \neq \emptyset$ , then the linear program  $(p, q, M)$  with  $p = r + M^T s$  has a unique solution  $\bar{x}$ . Furthermore,  $\bar{x} = -Xu$  for every  $u \in \mathcal{U}$ .

Proof: That the linear program has a solution  $\bar{x}$  follows from Proposition 2.6.

To show its uniqueness, let  $x$  be a solution to  $(p, q, M)$ . Then the proof of Proposition 2.6 shows that, in fact,

$$r^T x = s^T (q + Mx) = 0.$$



If  $I = \{i : r_i > 0\}$  and  $J = \{1, \dots, n\} \setminus I$ , then  $x$  must necessarily satisfy the following system of equations

$$(2.2) \quad \begin{pmatrix} 0 \\ q_J \end{pmatrix} + \begin{pmatrix} I & 0 \\ M_{JI} & M_{JJ} \end{pmatrix} \begin{pmatrix} x_I \\ x_J \end{pmatrix} = 0.$$

Here we have used the fact that  $s_J > 0$  by (M3)'. By the nondegeneracy of  $M$ , the system (2.2) has a unique solution. This establishes the uniqueness of  $\bar{x}$ . The last conclusion of the corollary is immediate.  $\square$

Remark. The matrix  $M$  in Example 2.4 is degenerate; nevertheless, the proof of the corollary shows that any solution to the linear program  $(p, q, M)$  must satisfy  $q + Mx = 0$ . Since  $M$  is nonsingular, the LP-solution

$x = \begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}$  is unique. This solution was also obtained in [7] by actually

applying the simplex method to the linear program. Its uniqueness was not proven there.

The results above establish the relationship between vectors in  $\mathcal{U}$ , solutions to the linear complementarity problem  $(q, M)$  and solutions to the linear program  $(p, q, M)$ . In the sequel, we divide our discussion into cases. In each case, we shall construct a polyhedral set having a least (or greatest) element and demonstrate how a solution to the linear complementarity problem can be generated from this element. In the analysis, we shall need two fundamental results from lattice theory. These are stated in Propositions 2.9 and 2.10. Their proofs are easy and are omitted.

Definition 2.8. Let  $S$  be a subset of  $R^n$ . Then  $S$  is said to be a meet (join) semi-sublattice (of  $R^n$ ) if for every  $s, t \in S$ , the meet (join) of  $s$  and  $t$ , defined as the vector  $u = (u_i)$  where  $u_i = \min(s_i, t_i)$  ( $\max(s_i, t_i)$ ) for every  $i$ , belongs to  $S$ . The set  $S$  is bounded below (above) if there exists a vector  $s^* \in R^n$  such that  $s \geq (\leq) s^*$  for every  $s \in S$ .

Proposition 2.9. The following are equivalent:

- 1)  $L$  is a polyhedral meet (join) semi-sublattice of  $R^n$ .
- 2)  $L = \{s \in R^n : As \geq b\}$  for some matrix  $A$  and vector  $b$ , with  $A$  having at most one positive (negative) element in each row.

Proposition 2.10. Let  $L$  be a nonempty meet (join) semi-sublattice of  $R^n$ . If  $L$  is closed and bounded below (above), then  $L$  has a least (greatest) element. Furthermore, this element can be obtained by solving

$$(2.3) \quad \text{minimize (maximize) } f^T x \text{ subject to } x \in L$$

for any positive vector  $f$ .

We should also mention that a least (greatest) element of a meet (join) semi-sublattice  $L$  of  $R^n$  actually solves the program (2.3) for any non-negative vector  $f$  and it is the unique solution if  $f$  is positive.

In the theorem below, we describe a polyhedral set with a least element and show how this element can easily generate a solution to the problem  $(q, M)$  in the case  $\mathcal{U} \neq \emptyset$ .

Theorem 2.11. Define

$$\mathcal{J} = \bigcap_{i=1}^n \{u \in \mathbb{R}^n : u \leq 0, Xu \geq 0, Yu \geq 0, c_i u_i \geq -1\}.$$

Then  $\mathcal{J}$  has a least element  $\bar{u}$ . Furthermore, if  $\mathcal{U} \neq \emptyset$ , then  $c^T \bar{u} \leq -1$  and the vector  $\bar{x} = -X \begin{pmatrix} \bar{u} \\ c^T \bar{u} \end{pmatrix}$  solves the linear complementarity problem  $(q, M)$ .

Proof: Proposition 2.9 implies that  $\mathcal{J}$  is a meet semi-sublattice of  $\mathbb{R}^n$ .

It is obviously closed and nonempty because  $0 \in \mathcal{J}$ . We show that  $\mathcal{J}$  is bounded below. Let  $u \in \mathcal{J}$ , clearly  $u_i \geq -1/c_i$  for every  $i \in I$  where  $I = \{i : c_i > 0\}$ . Furthermore, letting  $J = \{i : c_i = 0\}$ , we have,

$$(2.4a) \quad X_{JJ} u_J \geq -X_{JI} u_I \geq -X_{JI} d_I$$

and

$$(2.4b) \quad Y_{JJ} u_J \geq -Y_{JI} u_I \geq -Y_{JI} d_I$$

where  $d = (d_i)$  is the vector with  $d_i = -1/c_i$  for  $i \in I$  and 0 otherwise. On the other hand, it follows from condition (M3) that

$$r_J^T X_{JJ} + s_J^T Y_{JJ} > -(r_I^T X_{IJ} + s_I^T Y_{IJ}) \geq 0,$$

i.e. the Z-matrices  $X_{JJ}$  and  $Y_{JJ}$  satisfy condition (C2). Therefore by

Lemma 1.2, the matrix  $\begin{pmatrix} X_{JJ} \\ Y_{JJ} \end{pmatrix}$  contains a complementary submatrix which

is Minkowski. This fact, together with inequalities (2.4a) and (2.4b)

implies that  $u_J$  is bounded below by some constant vector. Hence  $\mathcal{J}$  is

bounded below. Therefore it has a least element  $\bar{u}$  by Proposition 2.10.

Clearly  $-u \in \mathfrak{J}$  if  $u \in \mathcal{U}$ . Thus,  $\bar{u} \leq -u$  for every  $u \in \mathcal{U}$ . Hence  $c^T \bar{u} \leq -1$  if  $\mathcal{U} \neq \emptyset$ . It is then clear that  $\frac{\bar{u}}{c^T \bar{u}} \in \mathcal{U}$ . Therefore the last

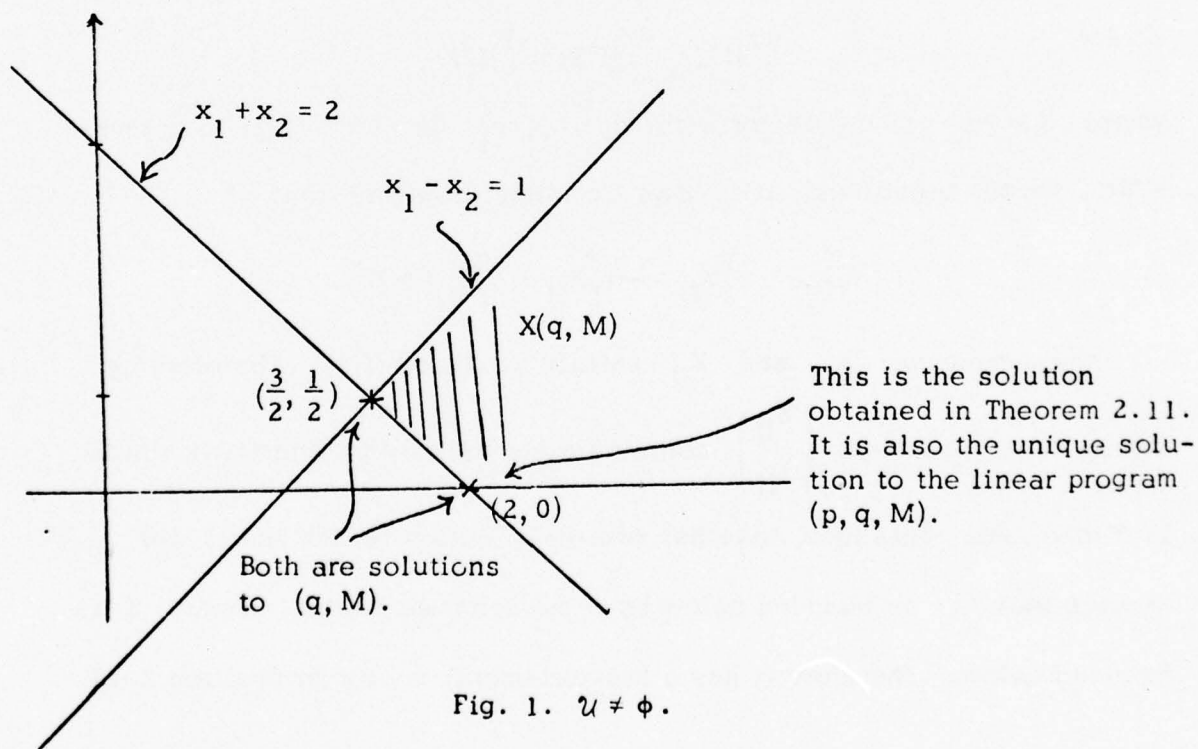
assertion of the theorem is an immediate consequence of Proposition 2.5.  $\square$

Remark. The assumption  $\mathcal{U} \neq \emptyset$  is essential in order for the vector  $\bar{x}$  in the theorem to be well-defined, because otherwise,  $c^T \bar{u}$  might be zero (e.g.  $c = 0$ ).

Example 2.12. Consider the problem  $(q, M)$  in Example 2.2. We have  $\mathcal{U} = \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}$  and

$$\mathfrak{J} = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2 : u_1 = 0 \text{ and } -2 \leq u_2 \leq 0 \right\}.$$

The least element of  $\mathfrak{J}$  is  $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ .





Combining Theorem 2.11 with Corollary 2.7, we conclude that if  $M$  is nondegenerate and if  $\mathcal{U} \neq \phi$ , then the (unique) solution to the linear program  $(p, q, M)$  is given by the vector  $\bar{x} = -X \frac{\bar{u}}{c^T \bar{u}}$  where  $\bar{u}$  is the least element of  $\mathcal{U}$ . This result provides a least-element interpretation to the solution of the linear program introduced by Mangasarian.

2.3. The case where  $\mathcal{U} = \phi$  and  $X$  is nonsingular. Having completed our discussion of the case  $\mathcal{U} \neq \phi$ , we proceed to investigate the case  $\mathcal{U} = \phi$ . We establish the following important lemma.

Lemma 2.13. If  $\mathcal{U} = \phi$ , then the following implication holds

$$(2.5) \quad \left. \begin{array}{l} u \geq 0 \\ Xu \leq 0 \\ Yu \leq 0 \end{array} \right\} \Rightarrow u = 0.$$

Proof: Suppose there exists a nonzero vector  $u$  such that  $u \geq 0$ ,  $Xu \leq 0$  and  $Yu \leq 0$ . Then we must have  $c^T u = 0$  because otherwise  $\frac{u}{c^T u} \in \mathcal{U}$  contradicting the assumption. Thus,  $Cu = 0$ . Hence,

$$\begin{aligned} 0 &< [r^T(X + C) + s^T(Y + C)]u \\ &\leq (r^T + s^T)Cu = 0 \end{aligned}$$

which is a contradiction.  $\square$

In fact, the converse of Lemma 2.13 holds, namely, if (2.5) holds, then  $\mathcal{U} = \phi$ . Now, by Proposition 1.1, condition (2.5) is an equivalent formulation of condition (C2). Therefore, adding the assumption  $\mathcal{U} = \phi$  to (M1) - (M3) is actually equivalent to replacing conditions (M2) and (M3) by the single stronger condition (C2).

Corollary 2.14. Let  $\mathcal{U} = \phi$ . Then

$$u \neq 0, Xu = 0 \Rightarrow c^T u \neq 0.$$

Proof: Indeed, if there exists a nonzero vector  $u$  such that  $Xu = 0$  and  $c^T u = 0$ . Then it follows from (M1) that  $Yu = 0$ . But this is impossible because by Lemma 2.13 above and Lemma 1.2, the matrix  $\begin{pmatrix} X \\ Y \end{pmatrix}$  contains a nonsingular complementary submatrix; this latter fact implies  $u = 0$ , contradicting the assumption.  $\square$

We have given an example earlier illustrating that there can be singular matrix  $X$  satisfying conditions (M1) - (M3). The proposition below shows that if  $\mathcal{U} = \phi$ , such matrices have rank at least  $n - 1$ .

Proposition 2.15. If  $\mathcal{U} = \phi$ , then  $\text{rank}(X) \geq n - 1$ .

Proof: Suppose  $\text{rank}(X) < n - 1$ . Let  $u^1$  and  $u^2$  be two linearly independent vectors such that  $Xu^1 = Xu^2 = 0$  and  $c^T u^1 = c^T u^2 = 1$ .

The existence of these vectors follows from the assumption and Corollary 2.14.

We then have

$$Yu^1 + q(c^T u^1) = Yu^2 + q(c^T u^2) = 0.$$

Thus,

$$Y(u^1 - u^2) = 0.$$

But we also have

$$X(u^1 - u^2) = 0.$$

Therefore,  $u^1 = u^2$ . This contradicts the linear independence of  $u^1$  and  $u^2$ .  $\square$

It is clear that there are instances when  $X$  is nonsingular, e.g.

$c = 0$ . The example below indicates that the case where  $\text{rank}(X) = n - 1$

is also possible even if  $\nu = \phi$ . Later in our discussion, we shall provide necessary and sufficient conditions for  $X$  to be nonsingular (see Theorem 2.22).

Example 2.16. Let  $q = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$  and  $M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ . The problem  $(q, M)$  has a solution  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Now let  $X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $c = e$ ,  $r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then  $MX = Y + qc^T$  and  $r^T X + s^T Y > 0$ . Therefore  $\nu = \phi$ . Nevertheless,  $X$  is of rank 1.  $\square$

In the sequel, we shall first establish the least-element results for the case where  $X$  is nonsingular. Then we shall prove two characterization theorems having to do with the nonsingularity of  $X$ . Finally, we shall investigate the last case, namely, where  $X$  has rank  $n - 1$ .

If  $X$  is nonsingular, then the linear complementarity problem  $(q, M)$  is equivalent to the following complementarity problem  $(LECP_1)$ :

$$q(1 + c^T u) + Yu \geq 0, Xu \geq 0 \text{ and } [q(1 + c^T u) + Yu]^T Xu = 0.$$

The equivalence is based on the transformation  $x = Xu$ . It is clear that  $x$  is a (feasible) solution to the problem  $(q, M)$  if and only if  $u = X^{-1}x$  is a (feasible) solution to the problem  $(LECP_1)$ . Let  $\mathcal{F}_1$  denote the feasible set of the latter problem, i.e.

$$\mathcal{F}_1 = \{u \in R^n : q(1 + c^T u) + Yu \geq 0, Xu \geq 0\}.$$

Proposition 2.17. If  $\nu = \phi$ , then  $1 + c^T u \neq 0$  for every  $u \in \mathcal{F}_1$ .

Proof: Suppose there is a  $u \in \mathcal{F}_1$  such that  $1 + c^T u = 0$ . Then it follows that  $Yu \geq 0, Xu \geq 0$ . Since the matrix  $\begin{pmatrix} X \\ Y \end{pmatrix}$  contains a Minkowski complementary submatrix, we must have  $u \geq 0$  contradicting the assumption that  $1 + c^T u = 0$ .  $\square$

By the convexity of  $\mathfrak{F}_1$  it follows that one and only one of the following two statements must hold:

$$(2.6a) \quad 1 + c^T u > 0 \quad \text{for every } u \in \mathfrak{F}_1$$

$$(2.6b) \quad 1 + c^T u < 0 \quad \text{for every } u \in \mathfrak{F}_1.$$

Lemma 2.18. Let  $\mathcal{V} = \phi$  and  $\mathfrak{F}_1 \neq \phi$ . Suppose (2.6a) ((2.6b)) holds. Let

$$\mathcal{L}_1 = \{v \in R^n : q + Yv \geq (\leq) 0, Xv \geq (\leq) 0\}.$$

Then  $\mathcal{L}_1$  has a least (greatest) element  $\bar{v}$  satisfying  $c^T \bar{v} < (>) 1$ .

Furthermore,  $(q + Y\bar{v})^T (X\bar{v}) = 0$ .

Proof: Proposition 2.9 implies that  $\mathcal{L}_1$  is a meet (join) semi-sublattice of  $R^n$ . It is obviously closed. The fact that  $\begin{pmatrix} X \\ Y \end{pmatrix}$  contains a Minkowski complementary submatrix implies that  $\mathcal{L}_1$  is bounded below (above). We show that it is nonempty. Let  $u \in \mathfrak{F}_1$  satisfy  $1 + c^T u > (<) 0$ , then it is not hard to see that the vector

$$(2.7) \quad v = \frac{u}{1 + c^T u}$$

belongs to  $\mathcal{L}_1$ . By Proposition 2.10,  $\mathcal{L}_1$  has a least (greatest) element  $\bar{v}$ .

That  $\bar{v}$  satisfies  $c^T \bar{v} < (>) 1$  is clear because  $\bar{v} \leq (\geq) v$  and  $c^T v < (>) 1$

where  $v$  is the vector defined in (2.7). To show that  $\bar{v}$  satisfies the

complementarity property, we refer the reader to the proof of Lemma 3.10

in [2].  $\square$

Remark. The polyhedral set  $\mathcal{L}_1$  does not depend on the vector  $c$ . If

(2.6a) holds, this set  $\mathcal{L}_1$  is precisely the set  $V$  mentioned in the introduction.



The following result generalizes Theorem 1.4 pertaining to the problem  $(q, M)$  with  $M \in \mathbb{C}$  which corresponds to the case  $c = 0$ .

Theorem 2.19. Let  $\mathcal{U} = \phi$ . Suppose  $X$  is nonsingular and the problem  $(q, M)$  is feasible. Then the vector

$$(2.8) \quad \bar{x} = X \left( \frac{\bar{v}}{1 - c^T \bar{v}} \right)$$

where  $\bar{v}$  is the vector obtained in Lemma 2.18, is a solution to the linear complementarity problem  $(q, M)$ .

Proof: In fact, if  $x \in X(q, M)$ , then  $u = X^{-1}x \in \mathcal{F}_1$ . Therefore the vector  $\bar{v}$  is well-defined and so is  $\bar{u} = \frac{\bar{v}}{1 - c^T \bar{v}}$ . It is easy to show that  $\bar{u}$  solves the problem  $(LECP)_1$ . Therefore by the equivalence of the problems  $(q, M)$  and  $(LECP)_1$  mentioned earlier, it follows that the vector  $\bar{x}$  defined in (2.8) solves the problem  $(q, M)$ .  $\square$

Theorem 2.19 shows how, under the assumptions in the theorem, a solution to the linear complementarity problem  $(q, M)$  can be generated from the least (or greatest) element of the polyhedral set  $\mathcal{F}_1$ . One notices that in some instances, the linear complementarity problem has a solution generated from the greatest element of a polyhedral set. This may seem, at first glance, somewhat inconsistent with the title "A least-element theory ..." of the report and with the phrases "least-element solutions", "least-element characterization", etc. ... which have been used throughout the report. However, it is not hard to see that such

greatest-element results can always be changed into least-element results by a very trivial modification, namely, by considering the negative of the polyhedral sets having the greatest elements. We have not made this change of variables in Lemma 2.18, and will not do so later because we want to present the results in their most natural format.

In the sequel, we establish a relationship between the vector  $\bar{x}$  generated in Theorem 2.19 and the solution(s) to the linear program  $(p, q, M)$  with  $p = r + M^T s$ . Specifically, we show that  $\bar{x}$  solves any such linear program; moreover, it is the unique solution if the vectors  $r$  and  $s$  satisfy the stronger condition (C2).

If  $X$  is nonsingular, then under the nonsingular transformation,  $x = Xu$ , the linear program  $(p, q, M)$  is equivalent to

$$(2.9) \quad \text{minimize } (p^T X)u \quad \text{subject to } q + MXu \geq 0 \quad \text{and } Xu \geq 0.$$

Noting that

$$p^T X = (r^T + s^T M)X = (r^T X + s^T Y) + (s^T q)c^T$$

we may write (2.9) as

$$(2.10) \quad \text{minimize } (r^T X + s^T Y)u + s^T q(1 + c^T u) \quad \text{subject to } u \in \mathcal{F}_1.$$

In order to simplify the following discussion, we assume that (2.6a) holds. Since  $\bar{v}$  is the least element of  $\mathcal{L}_1$ , it solves the linear program

$$(2.11) \quad \text{minimize } f^T v \quad \text{subject to } v \in \mathcal{L}_1$$

for any nonnegative vector  $f$ , and it is the unique solution to (2.11) for any positive vector  $f$ . It is then easy to deduce that the vector

$$(2.12) \quad \bar{u} = \frac{\bar{v}}{1 - c^T \bar{v}} \quad (\text{or equivalently, } \bar{v} = \frac{\bar{u}}{1 + c^T \bar{u}})$$

solves the (fractional) program:

$$(2.13) \quad \text{minimize } \frac{f^T u}{1 + c^T u} \quad \text{subject to } u \in \mathcal{F}_1$$

for any nonnegative vector  $f$ , and it is the unique solution to (2.13) for any positive vector  $f$ .

Theorem 2.20. Let  $\mathcal{U} = \phi$ . Suppose  $X$  is nonsingular and  $(q, M)$  is feasible. Then the vector  $\bar{x}$  (defined in (2.8)) solves the linear program  $(p, q, M)$  with  $p = r + M^T s$ . Moreover, it is the unique solution to any such linear program with  $r^T X + s^T Y > 0$ .

Remark. This theorem is valid no matter which one of the two inequalities (2.6a) and (2.6b) is valid. To simplify the proof below, we continue to assume (2.6a) holds.

Proof of the theorem: It suffices to show that the vector  $\bar{u}$  (defined in (2.12)) solves (2.10) and  $\bar{u}$  is the unique solution to (2.10) if  $r^T X + s^T Y > 0$ . Let  $u \in \mathcal{F}_1$ , then  $\frac{u}{1 + c^T u} \in \mathcal{S}_1$ . Thus

$$(r^T X + s^T Y) \frac{\bar{u}}{1 + c^T \bar{u}} \leq (r^T X + s^T Y) \frac{u}{1 + c^T u}$$

which implies

$$(2.14) \quad (r^T X + s^T Y) \bar{u} + s^T q (1 + c^T \bar{u}) \leq [(r^T X + s^T Y) u + s^T q (1 + c^T u)] \frac{1 + c^T \bar{u}}{1 + c^T u}.$$

Furthermore, we have

$$\bar{v} = \frac{\bar{u}}{1 + c^T \bar{u}} \leq \frac{u}{1 + c^T u}.$$

Hence,

$$\frac{1 + c^T \bar{u}}{1 + c^T u} \leq 1.$$

On the other hand,

$$(r^T X + s^T Y) u + s^T q(1 + c^T u) = r^T (Xu) + s^T (q + (Y + qc^T)u) \geq 0.$$

Therefore, for  $u \in \mathcal{F}_1$ ,

$$(r^T X + s^T Y) \bar{u} + s^T q(1 + c^T \bar{u}) \leq (r^T X + s^T Y) u + s^T q(1 + c^T u)$$

i.e.  $\bar{u}$  solves (2.10).

Conversely, if  $u$  solves (2.10), then (2.14) must hold as equality.

Therefore, we have

$$(r^T X + s^T Y) \frac{\bar{u}}{1 + c^T \bar{u}} = (r^T X + s^T Y) \frac{u}{1 + c^T u}$$

i.e.  $u$  also solves (2.13). Now if  $r^T X + s^T Y > 0$ , it follows from the uniqueness of  $\bar{u}$  that  $u = \bar{u}$ .  $\square$

Example 2.21. Let  $q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$  and  $M = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$ . The problem  $(q, M)$  has solutions  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . If  $X = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ ,  $Y = \begin{pmatrix} \frac{1}{2} & 0 \\ -1 & 4 \end{pmatrix}$ ,  $r = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ,  $s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $c = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ , then  $MX = Y + qc^T$  and  $r^T X + s^T Y > 0$ . It is not hard to show that  $M \notin \mathcal{C}$ .



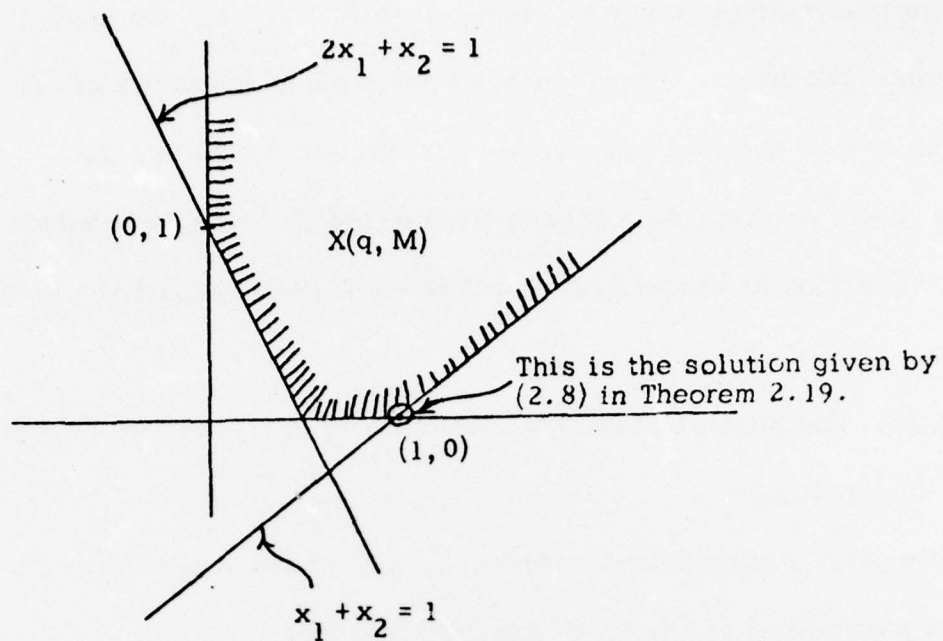


Fig. 2.  $\mathcal{U} = \phi$  and  $X$  nonsingular.

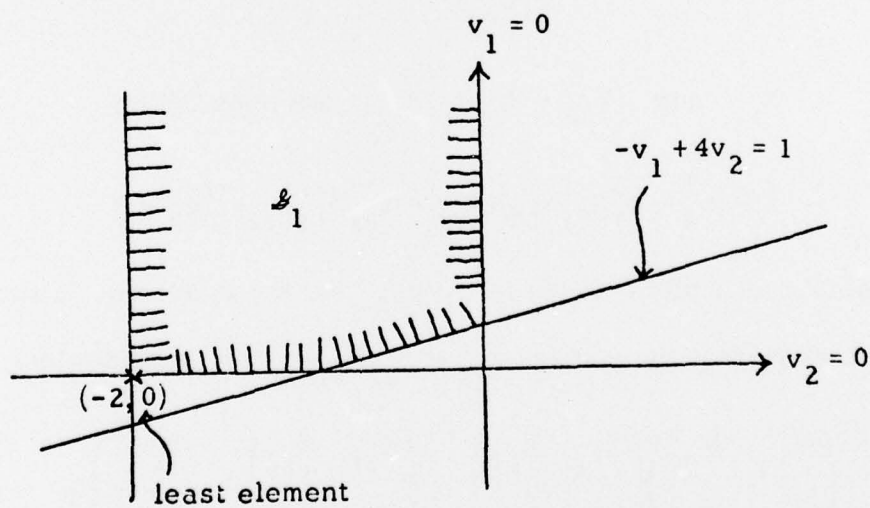


Fig. 3. The set  $\mathcal{S}_1$ .

Before proceeding to study the case  $\text{rank}(X) = n - 1$ , we prove two theorems. The first of which characterizes the nonsingularity of  $X$ ; whereas the second provides necessary and sufficient conditions for (2.6a) and (2.6b) to hold. As a consequence of the theorems, we deduce that the assumptions in Lemma 2.18 and Theorem 2.19 are actually equivalent.

Theorem 2.22. The following two conditions (2.15) and (2.16) are equivalent:

(2.15)  $X$  is nonsingular

(2.16) there exist a principal rearrangement, with permutation matrix  $P$ , and partitioning of  $X$ ,  $Y$ ,  $q$  and  $c$  such that

$$(2.16a) \quad P^T X P = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad P^T Y P = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

$$(2.16b) \quad P^T q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad P^T c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$(2.16c) \quad X_{11} \text{ and } (Y_{22} - Y_{21} X_{11}^{-1} X_{12}) \text{ are nonsingular}$$

$$(2.16d) \quad 1 + (c_2^T - c_1^T X_{11}^{-1} X_{12})(Y_{22} - Y_{21} X_{11}^{-1} X_{12})^{-1} q_2 \neq 0.$$

Proof: That (2.15) implies (2.16) is trivial. For the converse, it suffices

(see [1]) to show that the matrix  $X_{22} - X_{21} X_{11}^{-1} X_{12}$  is nonsingular. We have

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} + \begin{pmatrix} q_1 c^T \\ q_2 c^T \end{pmatrix} P.$$

Solving for  $M_{21}$  in

$$M_{21}X_{11} + M_{22}X_{21} = Y_{21} + q_2 c_1^T$$

gives

$$M_{21} = (Y_{21} - M_{22}X_{21})X_{11}^{-1} + q_2 c_1^T X_{11}^{-1}.$$

Substituting this latter equality in

$$M_{21}X_{12} + M_{22}X_{22} = Y_{22} + q_2 c_2^T$$

yields

$$\begin{aligned} M_{22}(X_{22} - X_{21}X_{11}^{-1}X_{12}) &= (Y_{22} - Y_{21}X_{11}^{-1}X_{12}) + q_2(c_2^T - c_1^T X_{11}^{-1}X_{12}) \\ &= (Y_{22} - Y_{21}X_{11}^{-1}X_{12})[I + (Y_{22} - Y_{21}X_{11}^{-1}X_{12})^{-1}q_2(c_2^T - c_1^T X_{11}^{-1}X_{12})]. \end{aligned}$$

Condition (2.16d) implies

$$\begin{aligned} \det[I + (Y_{22} - Y_{21}X_{11}^{-1}X_{12})^{-1}q_2(c_2^T - c_1^T X_{11}^{-1}X_{12})] \\ = 1 + (c_2^T - c_1^T X_{11}^{-1}X_{12})(Y_{22} - Y_{21}X_{11}^{-1}X_{12})q_2 \neq 0. \end{aligned}$$

This establishes the nonsingularity of  $X_{22} - X_{21}X_{11}^{-1}X_{12}$ .  $\square$

Remark. The argument used above is a generalization of the one in Theorem 1.3 for the case  $M \in \mathbb{C}$ . In the proof, we have used the condition (M1) only. So in fact, the theorem provides necessary and sufficient conditions for the matrix  $X$  (not necessarily Z-matrix) satisfying  $MX = Y + qc^T$  for some matrices  $M$  and  $Y$  (not necessarily Z-matrix) and some vectors  $q$  and  $c$  (not necessarily non-negative), to be nonsingular.

Theorem 2.23. Let  $\mathcal{U} = \phi$ . Suppose  $\mathfrak{F}_1 \neq \phi$ . Then (2.6a) ((2.6b)) holds if and only if there exist a principal rearrangement with permutation matrix  $P$ , and partitioning of  $X$ ,  $Y$ ,  $q$  and  $c$  such that

$$(2.17a) \quad P^T X P = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \quad P^T Y P = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

$$(2.17b) \quad P^T q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad P^T c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$(2.17c) \quad \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \in K$$

$$(2.17d) \quad 1 + (c_2^T - c_1^T X_{11}^{-1} X_{12})(Y_{22} - Y_{21} X_{11}^{-1} X_{12}) q_2 > (<) 0.$$

Proof: "Sufficiency". Let  $u \in \mathfrak{F}_1$ . Then we have

$$\begin{pmatrix} 0 \\ q_2(1 + c^T u) \end{pmatrix} + \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \geq 0.$$

Condition (2.17c) implies that  $\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix}^{-1} \geq 0$ ; thus

$$(2.18) \quad \begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ q_2 \end{pmatrix} (1 + c^T u) + u \geq 0.$$

Let  $t = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$  satisfy

$$\begin{pmatrix} X_{11} & X_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ q_2 \end{pmatrix}.$$



Then,

$$t_1 = -X_{11}^{-1}X_{12}t_2$$

and,

$$t_2 = (Y_{22} - Y_{21}X_{11}^{-1}X_{12})^{-1}q_2.$$

We can write (2.18) as

$$t(1 + c^T u) + u \geq 0$$

which implies

$$(2.20) \quad (1 + c^T t)(1 + c^T u) \geq 1.$$

Now, by condition (2.17d)

$$(2.19) \quad 1 + c^T t = 1 + (c_2^T - c_1^T X_{11}^{-1} X_{12})t_2 \\ = 1 + (c_2^T - c_1^T X_{11}^{-1} X_{12})(Y_{22} - Y_{21}X_{11}^{-1}X_{12})^{-1}q_2 > (<) 0.$$

Therefore  $1 + c^T u > (<) 0$  for every  $u \in \mathcal{F}_1$ , i.e. (2.6a) ((2.6b)) holds.

"Necessity". The existence of the permutation and partitioning such that conditions (2.17a) - (2.17c) are satisfied is an immediate consequence of the assumption  $\mathcal{U} = \phi$  and Lemma 1.2. So it remains to verify (2.17d).

The deduction above shows that (2.20) is valid for every  $u \in \mathcal{F}_1$ . Now if  $\mathcal{F}_1 \neq \phi$  and (2.6a) ((2.6b)) holds, then it follows from (2.20) that  $1 + c^T t > (<) 0$ , or equivalently, (2.17d) holds.  $\square$

We have seen that if  $X$  is nonsingular and if the problem  $(q, M)$  is feasible, then  $\mathcal{F}_1 \neq \phi$ . Combining Theorems 2.22 and 2.23, we conclude that if  $\mathcal{U} = \phi$  and if  $\mathcal{F}_1 \neq \phi$  then  $(q, M)$  is feasible (in fact, has a solution) and  $X$  must be nonsingular.

Corollary 2.24. If  $\mathcal{U} = \phi$ , the following are equivalent

(2.21)  $X$  is nonsingular and  $(q, M)$  has a solution (given by (2.8))

(2.22)  $\mathfrak{F}_1 \neq \phi$ .

2.4. The case where  $\mathcal{U} = \phi$  and  $X$  is singular. In the rest of this paper, we investigate the remaining case, namely  $\mathcal{U} = \phi$  and  $\text{rank}(X) = n - 1$ . Let  $u^0$  be the (unique) vector satisfying

$$Xu^0 = 0 \text{ and } c^T u^0 = 1.$$

The existence of such a vector follows from Corollary 2.14. Then, it follows that

$$(2.23) \quad q = -Yu^0.$$

Let  $u^1$  be a nonzero vector satisfying  $X^T u^1 = 0$ . It follows from elementary linear algebra that, for any  $x \in R^n$ , there exist  $u \in R^n$  and  $\alpha \in R^1$  such that

$$(2.24) \quad x = Xu + \alpha u^1.$$

Under such a representation, we have, by (2.23)

$$\begin{aligned} q + Mx &= q + MXu + \alpha Mu^1 \\ &= q(1 + c^T u) + Yu + \alpha Mu^1 \\ &= Y[u - (1 + c^T u)u^0] + \alpha Mu^1. \end{aligned}$$

Proposition 2.25. Under the identification (2.24), the linear complementarity problem  $(q, M)$  is equivalent to the problem  $(LECP_2)$  of finding a vector  $u \in R^n$  and a scalar  $\alpha$  such that

$$\alpha M u^1 + Y[u - (1 + c^T u) u^0] \geq 0$$

$$\alpha u^1 + X[u - (1 + c^T u) u^0] \geq 0$$

$$[\alpha M u^1 + Y(u - (1 + c^T u) u^0)]^T [\alpha u^1 + X(u - (1 + c^T u) u^0)] = 0.$$

Let  $\mathcal{F}_2$  denote the feasible set of (LECP<sub>2</sub>), i.e.

$$\mathcal{F}_2 = \left\{ \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1} : \alpha M u^1 + Y(u - (1 + c^T u) u^0) \geq 0, \alpha u^1 + X(u - (1 + c^T u) u^0) \geq 0 \right\}.$$

Proposition 2.26. If  $\mathcal{U} = \emptyset$ , then  $\alpha \neq 0$  for every  $\begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{F}_2$ .

Proof: Suppose there exists  $\begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{F}_2$  such that  $\alpha = 0$ . Then

$$Y[u - (1 + c^T u) u^0] \geq 0$$

$$X[u - (1 + c^T u) u^0] \geq 0.$$

Hence, by Lemma 1.2, it follows that  $u - (1 + c^T u) u^0 \geq 0$ . Thus

$$0 \leq c^T [u - (1 + c^T u) u^0] = -1$$

which is a contradiction.  $\square$

Remark. An equivalent formulation of the conclusion in Proposition 2.26 can be stated as  $X(q, M) \cap \mathcal{R}(X) = \emptyset$ .

By the convexity of  $\mathcal{F}_2$ , it follows that one and only one of the following two statements must hold:

$$(2.25a) \quad \alpha > 0 \text{ for every } \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{F}_2$$

$$(2.25b) \quad \alpha < 0 \text{ for every } \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{F}_2.$$

Lemma 2.27. Let  $\mathcal{U} = \emptyset$  and  $\mathcal{F}_2 \neq \emptyset$ . Suppose (2.25a) ((2.25b)) holds.

Define

$$\mathcal{L}_2 = \{v \in \mathbb{R}^n : M u^1 + Yv \geq (\leq) 0, u^1 + Xv \geq (\leq) 0\}.$$

Then  $\mathcal{L}_2$  has a least (greatest) element  $\bar{v}$  satisfying  $c^T \bar{v} < (>) 0$ .

Furthermore,  $(Mu^1 + Y\bar{v})^T(u^1 + X\bar{v}) = 0$ .

Proof: If  $\begin{pmatrix} u \\ \alpha \end{pmatrix} \in \mathcal{F}_2$ , then  $v = \frac{1}{\alpha}(u - (1 + c^T u)u^0) \in \mathcal{L}_2$  and  $c^T v = -\frac{1}{\alpha}$ .

The rest of proof proceeds in the same way as in the proof of Lemma 2.18.  $\square$

Theorem 2.28. Let  $\mathcal{U} = \phi$ . Suppose  $\text{rank}(X) = n - 1$  and the problem  $(q, M)$  is feasible. Then the vector

$$(2.26) \quad \bar{x} = -\frac{1}{c^T \bar{v}}(X\bar{v} + u^1)$$

where  $\bar{v}$  is the vector obtained in Lemma 2.27, is a solution to the linear complementarity problem  $(q, M)$ .

Proof: The assumptions in Lemma 2.27 are satisfied. Therefore,  $\bar{x}$  is well-defined. It suffices to show that  $\bar{u} = -\frac{\bar{v}}{c^T \bar{v}}$  and  $\bar{\alpha} = -\frac{1}{c^T \bar{v}}$  solves the problem  $(LECP_2)$ . Noting that  $1 + c^T \bar{u} = 0$ , we can easily deduce that  $\begin{pmatrix} \bar{u} \\ \bar{\alpha} \end{pmatrix}$  indeed solves  $(LECP_2)$ . This establishes the theorem.  $\square$

Having demonstrated how, under the assumptions of Theorem 2.28, a solution to the linear complementarity problem can be generated from the least (or greatest) element of the polyhedral set  $\mathcal{L}_2$ , we next establish a relationship between this solution  $\bar{x}$  and the solution(s) to the linear program  $(p, q, M)$  with  $p = r + M^T s$ .

Theorem 2.29. Let  $\mathcal{U} = \phi$ . Suppose  $\text{rank}(X) = n - 1$  and  $(q, M)$  is feasible. Then the vector  $\bar{x}$  (defined in (2.26)) solves the linear program  $(p, q, M)$  with  $p = r + M^T s$ . Moreover it is the unique solution to any such linear program with  $r^T X + s^T Y > 0$ .



Proof: Assume for simplicity, that (2.25a) holds. The argument which follows is similar to that of Theorem 2.20. Let  $x = Xu + \alpha u^1 \in X(q, M)$ , then  $v = \frac{1}{\alpha}(u - (1 + c^T u)u^0) \in \mathcal{L}_2$ . Hence we have

$$\bar{v} \leq \frac{1}{\alpha}(u - (1 + c^T u)u^0)$$

which implies

$$c^T \bar{v} \leq -\frac{1}{\alpha}.$$

Moreover, by (2.23),

$$\begin{aligned} (2.27) \quad (r^T X + s^T Y) \bar{v} &\leq \frac{1}{\alpha} (r^T X + s^T Y) (u - (1 + c^T u)u^0) \\ &= \frac{1}{\alpha} [(r^T X + s^T Y)u + s^T q(1 + c^T u)] . \end{aligned}$$

Furthermore,

$$r^T (Xu + \alpha u^1) + s^T [q + M(Xu + \alpha u^1)] \geq 0 .$$

Combining the above inequalities, we deduce,

$$\begin{aligned} p^T \bar{x} &= (r^T + s^T M) \left( -\frac{1}{c^T \bar{v}} (X\bar{v} + u^1) \right) \\ &= -\frac{1}{c^T \bar{v}} [(r^T X + s^T Y) \bar{v} + s^T q(c^T \bar{v}) + (r^T + s^T M)u^1] \\ &\leq -\frac{1}{c^T \bar{v}} \left[ \frac{1}{\alpha} \{ (r^T X + s^T Y)u + \alpha(r^T + s^T M)u^1 + s^T q(1 + c^T u) \} \right] - s^T q \\ &= \left( -\frac{1}{c^T \bar{v}} \right) \left( \frac{1}{\alpha} [r^T (Xu + \alpha u^1) + s^T (q + M(Xu + \alpha u^1))] \right) - s^T q \\ &\leq (r^T + s^T M)x = p^T x . \end{aligned}$$

Therefore  $\bar{x}$  solves the linear program  $(p, q, M)$ .

Conversely, if  $x$  is a solution to  $(p, q, M)$ , then we must have equality in (2.27). If  $r^T X + s^T Y > 0$ , then by the uniqueness of  $\bar{v}$ , it follows that  $\bar{v} = \frac{1}{\alpha}(u - (1 + c^T u)u^0)$ . Thus  $\alpha = -\frac{1}{c^T \bar{v}}$  and  $Xu = -\frac{1}{c^T \bar{v}} X\bar{v}$ . Therefore,  $x = Xu + \alpha u^1 = -\frac{1}{c^T \bar{v}}(X\bar{v} + u^1) = \bar{x}$ . This establishes the theorem.  $\square$

We conclude this paper by proving a result (Corollary 2.31) which strengthens Corollary 4 of Mangasarian's report [7] where it was shown that if  $n \geq 3$  and  $M$  is a positive matrix which is diagonally dominant column-by-column, that is,  $m_{jj} \geq \sum_{i \neq j} m_{ij}$ ,  $j = 1, \dots, n$ , then conditions (M1) - (M3) are satisfied (in fact for every  $n$ -vector  $q$ ).

Proposition 2.30. Let  $M$  be an  $n \times n$  matrix satisfying

$$(2.28a) \quad MX = Y \text{ for some } X, Y \in Z$$

$$(2.28b) \quad r^T X \geq 0 \text{ for some } r > 0$$

$$(2.28c) \quad Y = (Y_{ij}) \text{ is such that } Y_{ij} < 0 \text{ for every } j \neq i.$$

Then  $M \in \mathcal{C}$ .

Proof: Let  $\epsilon > 0$  be small enough such that  $Y + \epsilon M \in Z$ . Then

$$M(X + \epsilon I) = Y + \epsilon M \text{ and } X + \epsilon I \in K. \text{ Therefore } M \in \mathcal{C}. \quad \square$$

Corollary 2.31. Let  $n \geq 3$  and let  $M$  be a positive  $n \times n$  matrix

which satisfies either of the following conditions:

$$(2.29a) \quad \text{diagonal dominance column-by-column: } m_{jj} \geq \sum_{i \neq j} m_{ij}, \quad j = 1, \dots, n$$

$$(2.29b) \quad \text{diagonal dominance row-by-row: } m_{ii} \geq \sum_{j \neq i} m_{ij}, \quad i = 1, \dots, n.$$

Then  $M \in \mathcal{C}$ .

Proof: We may without loss of generality, assume that  $M$  has been normalized so that it has ones on the diagonal. Let  $M = I + F$  where  $F = (F_{ij})$ ,  $F_{ii} = 0$  and  $F_{ij} = m_{ij}$  for  $i \neq j$ . Define  $X = I - F$  and  $Y = MX = I - F^2$ . Clearly  $X$  and  $Y \in Z$ . Furthermore, it is easy to see that condition (2.28c) is satisfied. We show that (2.28b) is also satisfied. It is clearly satisfied if (2.29a) holds. Now, if (2.29b) is true, then  $X \in K_0$  (by Theorem (5.4) in [3]) and  $X$  is irreducible (see [8]). These two properties of  $X$  imply (2.28b) (by Theorem (5.8) in [3]). Consequently, by Proposition 2.30,  $M \in C$ .  $\square$

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TR 76-30: "A Least Element Theory of Solving Linear Complementarity Problems as Linear Programs",  
by Richard W. Cottle and Jong-Shi Pang

In a previous report, ~~the authors have~~ established a least-element interpretation to Mangasarian's theory of formulating some linear complementarity problems as linear programs. In the present report, ~~we~~ extend ~~our~~ <sup>the</sup> previous analysis to a more general class of linear complementarity problems investigated by Mangasarian. <sup>The</sup> Our purposes are (1) to demonstrate how solutions to these problems can be generated from least elements of polyhedral sets and (2) to investigate how these "least-element solutions" are related to the solutions obtained by the linear programming approach as proposed by Mangasarian.

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